LEARNING TARGET: I will solve nonlinear systems of equations both algebraically and graphically.

Review of Linear Systems

A system of two linear equations in two variables x and y, also called a *linear system*, consists of two equations that can be written in the form:

$$Ax + By = C$$
$$Dx + Ey = F$$

A **solution** of a system of linear equations in two variables in an ordered pair (x, y) that satisfies each equation. Solutions correspond to points where the graphs of the equations in a system intersect.

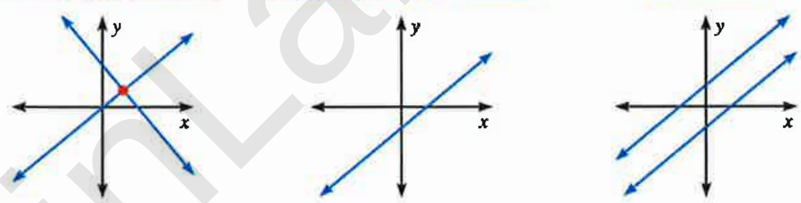
A system that has at least one solution is **consistent**. If a system has no solution, the system is **inconsistent**.

A system that has exactly one solution is **independent**, and a consistent system that has infinitely many systems is **dependent**.

Number of Solutions of a Linear System

The relationship between the graph of a linear system and the system's number of solutions is described below.

Exactly one solution | **Infinitely many solutions** | **No solution**



Lines intersect at one point; consistent and independent

Lines coincide; consistent and dependent

Lines are parallel; inconsistent

In addition to graphing, systems of linear equations can also be solved algebraically using the **substitution method** or **elimination method**.

Substitution Method:

1. Solve for a variable
2. Plug in result for variable in other equation
3. Solve for remaining variable.
4. Plug back in to get other coordinate

Elimination Method:

1. Line up variables (x + y ... = #)
2. Add or subtract to cancel one of the variables (may have to multiply through by a constant)
3. Solve for remaining variable.

PAIR ACTIVITY

Solve the following by graphing.

$y = x - 8$
$-x + 2y = -10$

$(6,-2)$

$\frac{2y}{2} = \frac{-10+x}{2}$

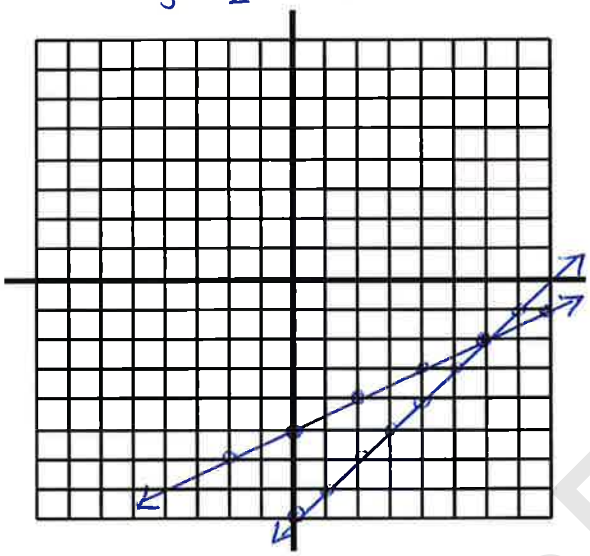
$y = \frac{1}{2}x - 5$

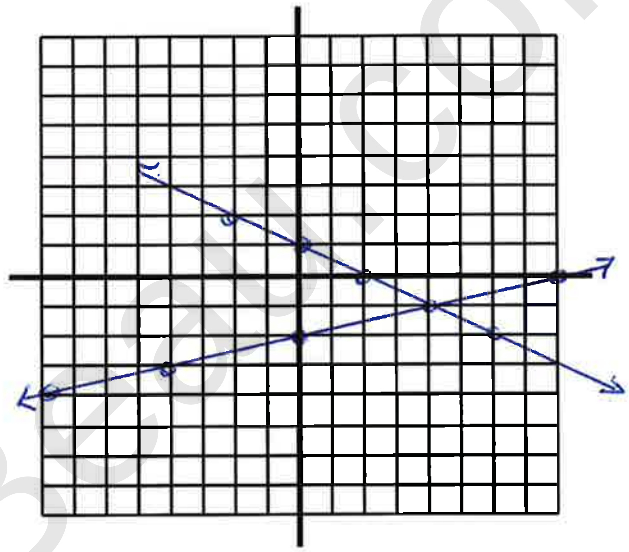
$x + 2y = 2$
$x - 4y = 8$

$y = -\frac{1}{2}x + 1$

$y = \frac{1}{4}x - 2$

$(4,-1)$





QUESTIONS

1. How can you determine by looking at the graph of a linear system if a solution exists?

   a solution exists if the two lines intersect

2. What would a graph of a system with no solution look like?

   The lines would be parallel

3. Solve the following system – how many solutions does it have?

   $$x - 7y = 6$$
   $$-3x + 21y = -18$$

   $x = 6 + 7y$

   $-3(6+7y) + 21y = -18$
   $-18 - 21y + 21y = -18$
   $-18 = -18$

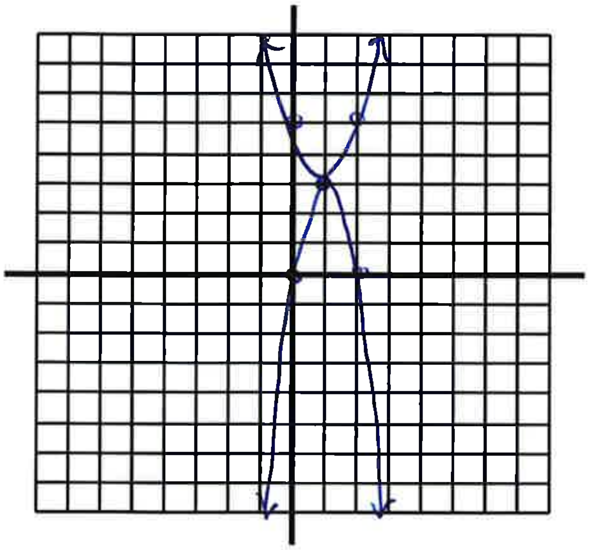
   infinate solutions

4. Describe what the graph of a system of equations with infinite solutions looks like.

   Infinate solutions occur when one equation is a multiple of the other.

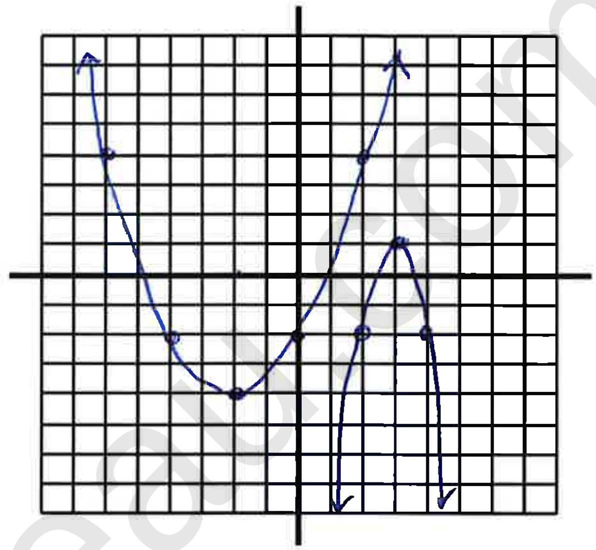
Given what you know – see if you can solve the following systems graphically.

$y = 2(x-1)^2 + 3$
$y = -3(x-1)^2 + 3$



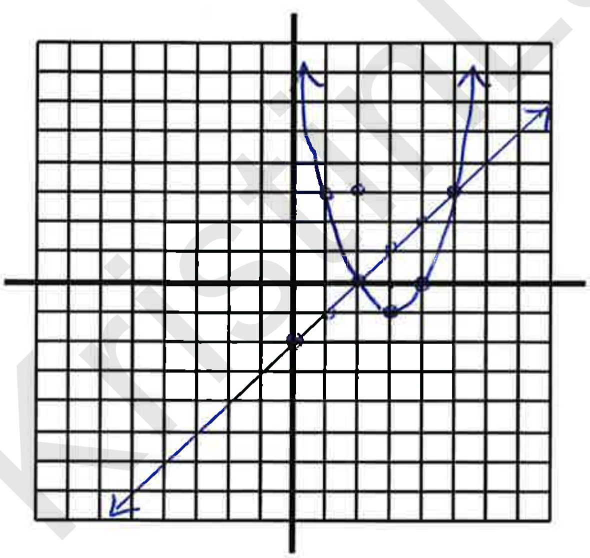
Solutions:
(1, 3)

$y = \frac{1}{2}(x+2)^2 - 4$
$y = -3(x-3)^2 + 1$



Solutions:
DNE

$y = (x-3)^2 - 1$
$y = x - 2$



Solutions:
(2,0) + (5,3)

Solve the following systems using substitution.

Algebra I

$2x + 5y = 7$
$x = -4y + 2$

$(6, -1)$

$2(-4y+2)+5y=7$

$-8y+4+5y=7$

$-3y+4=7$

$\quad -4 \quad -4$

$\frac{-3y}{-3} = \frac{3}{-3}$

$y=-1$

$x=-4(-1)+2$

$x=4+2$

$x=6$

CHECK

$12-5=7$

$7=7$ ✓

$6=4+2$

$6=6$ ✓

$x = -4y + 1$
$3x + 2y = -12$

$(-5, 3/2)$

$3(-4y+1)+2y=-12$

$-12y+3+2y=-12$

$-10y+3=-12$

$\quad -3 \quad -3$

$\frac{-10y}{-10} = \frac{-15}{-10}$

$y=\frac{3}{2}$

$x=-4(3/2)+1$

$x=-5$

CHECK

$-15+3=-12$

$-12=-12$ ✓

Algebra II

$x^2 + 2x + 5 = y$
$x^2 + 4x - 6 = y + 2$

$\left(\frac{13}{2}, \frac{241}{4}\right)$

$x^2+4x-6=x^2+2x+5+2$

$x^2+4x-6=x^2+2x+7$

$-x^2-2x-7 \quad -x^2-2x-7$

$2x-13=0$

$2x=13$

$x=13/2$

$\left(\frac{13}{2}\right)^2+2\left(\frac{13}{2}\right)+5=y$

$\frac{241}{4}=y$

$x + 1 = y$
$4x^2 - 3x + 2 = 2y$

$4x^2-3x+2=2(x+1)$

$4x^2-3x+2=2x+2$

$\quad -2x-2 \quad -2x-2$

$4x^2-5x=0$

$x(4x-5)$

$x=0 \quad x=5/4$

$0+1=y \quad \frac{5}{4}+1=y$

$1=y \quad 9/4=y$

Solve the following systems using elimination.

Algebra I

$-2(2x + y = 1)$
$4x + 2y = 2$

$-\cancel{4x} - \cancel{2y} = -2$
$\cancel{4x} + \cancel{2y} = 2$
$0 = 0$

infinate solutions

$3(x - y = 5)$
$-3x + 3y = 2$

$\cancel{3x} - \cancel{3y} = 15$
$-\cancel{3x} + \cancel{3y} = 2$
$0 = 17$

no solutions

Algebra II

$4x - y = 2$
$2x^2 - y = 0$

$(1, 2)$

$2x^2 + 0x - \cancel{y} = 0$
$+\ 0x^2 - 4x + \cancel{y} = -2$
$2x^2 - 4x = -2$
$2x^2 - 4x + 2$
$2(x^2 - 2x + 1)$
$2(x-1)(x-1)$
$x = 1$

$4(1) - y = 2$
$4 - y = 2$
$2 = y$

$x^2 + y^2 = 13$
$-(x^2 - y = 7)$

$(-3, 2)(2, -3)$
$(3, 2)$
$(-2, -3)$

$x^2 + y^2 + 0y = 13$
$+\ -x^2 + 0y^2 + y = -7$
$y^2 + y = 6$
$y^2 + y - 6$
$(y-2)(y+3)$
$y = 2 \quad y = -3$

$x^2 - 2 = 7$
$x^2 = 9$
$x = \pm 3$

$x^2 - (-3) = 7$
$x^2 + 3 = 7$
$x^2 = 4$
$x = 2$

DISCUSSION QUESTIONS

1. How do the solutions of a system and intersection points of their graphs relate?

The solutions of a system are the intersection points of their graphs.

2. If the point (3, 4) makes one equation true but the other equation in the system false – is the point a solution to the system? Why or why not?

No, the point is not a solution to the system. It must make both equations true to be a solution to the system.

3. Can you think of a real-world problem that could be solved using a system of equations?

- Knowing admission rates for children + adults and total revenue from ticket sales vs. how many were sold. We can solve this system to find out how many children + adults attended.

4. Compare and contrast the different methods to solve a system of equations.

Both elimination + substitution are algebraic + graphing is solving visually, but all will lead to the same answer.

SOLVING LINEAR SYSTEMS USING MATRICES

If you have a system of *linear equations* (the greatest power on any variable =1) then you can use matrices and **Cramer's rule** to solve. This method uses the **coefficient matrix** of a linear system to solve.

Linear System	**Coefficient Matrix**
$ax + by = e$ $cx + dy = f$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Each matrix has a real number associated with it called its **determinant**. The determinant of a 2 x 2 matrix and a 3 x 3 matrix is detailed below.

The Determinant of a Matrix

Determinant of a 2 × 2 Matrix

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

The determinant of a 2 × 2 matrix is the difference of the products of the elements on the diagonals.

Determinant of a 3 × 3 Matrix

STEP 1 **Repeat** the first two columns to the right of the determinant.

STEP 2 **Subtract** the sum of the **red products** from the sum of the **blue products**.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \left|\begin{array}{ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}\right. = (aei + bfg + cdh) - (gec + hfa + idb)$$

Cramer's Rule for a 2 × 2 System

Let A be the coefficient matrix of this linear system:

$$ax + by = e$$
$$cx + dy = f$$

If $\det A \neq 0$, then the system has exactly one solution. The solution is:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \quad \text{and} \quad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

Notice that the numerators for x and y are the determinants of the matrices formed by replacing the coefficients of x and y, respectively, with the column of constants.

Cramer's Rule for a 3 × 3 System

Let A be the coefficient matrix of the linear system shown below.

Linear System

$$\begin{aligned} ax + by + cz &= j \\ dx + ey + fz &= k \\ gx + hy + iz &= l \end{aligned}$$

Coefficient Matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

If $\det A \neq 0$, then the system has exactly one solution. The solution is:

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\det A}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\det A}, \quad \text{and} \quad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\det A}$$

Examples

Use Cramer's Rule to solve the following linear systems.

1. $9x + 4y = -6$
   $3x - 5y = -21$

$$\begin{bmatrix} 9 & 4 \\ 3 & -5 \end{bmatrix}$$

D: $9(-5) - 3(4)$
$-45 - 12$
$\boxed{-57}$

$$x = \frac{\begin{bmatrix} -6 & 4 \\ -21 & -5 \end{bmatrix}}{-57} \qquad \frac{30-(-84)}{-57} = \boxed{-2}$$

$$y = \frac{\begin{bmatrix} 9 & -6 \\ 3 & -21 \end{bmatrix}}{-57} = \frac{-189-(-18)}{-57} = \boxed{3}$$

$\boxed{-2, 3}$

2. $5x - y - 2z = -6$
   $x + 3y + 4z = 16$
   $2x - 4y + z = -15$

$\boxed{0, 4, 1}$

$$\begin{bmatrix} 5 & -1 & -2 \\ 1 & 3 & 4 \\ 2 & -4 & 1 \end{bmatrix} \begin{matrix} 5 & -1 \\ 1 & 3 \\ 2 & -4 \end{matrix}$$

$(15 + (-8) + 8) - (-12 + (-80) + (-1))$
$15 - (-93)$
D: $\boxed{108}$

$$x = \frac{\begin{bmatrix} -6 & -1 & -2 \\ 16 & 3 & 4 \\ -15 & -4 & 1 \end{bmatrix}}{108} = \frac{0}{108} = 0$$

$$y = \frac{\begin{bmatrix} 5 & -6 & -2 \\ 1 & 16 & 4 \\ 2 & -15 & 1 \end{bmatrix}}{108} = \frac{432}{108} = 4$$

$$z = \frac{\begin{bmatrix} 5 & -1 & -6 \\ 1 & 3 & 16 \\ 2 & -4 & -15 \end{bmatrix}}{108} = \frac{108}{108} = 1$$

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Systems of Equations Problem Set

Solve each system algebraically using either the substitution or elimination method. Then graph each system to verify your solution.

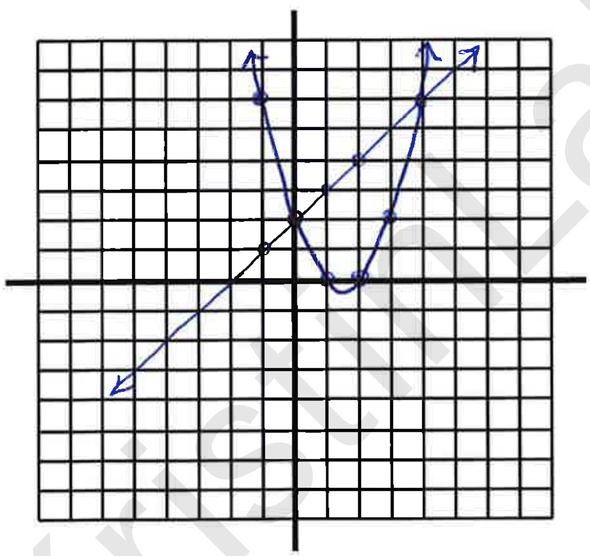
1. $y = x + 2$
   $y = x^2 - 3x + 2$

$$x + 2 = x^2 - 3x + 2$$
$$-x - 2 \qquad -x - 2$$

$$y = x^2 - 4x$$

$$x(x-4)$$

$$\boxed{\begin{array}{ll} x=0 & x=4 \\ y=2 & y=6 \end{array}}$$



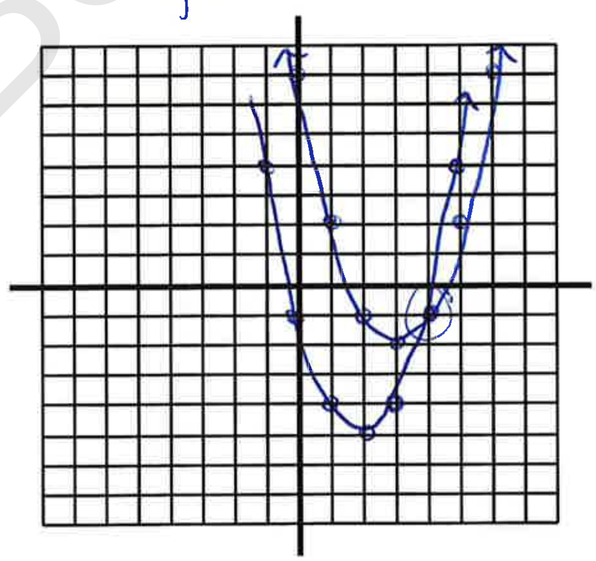
Solutions:
$(0,2)$ + $(4,6)$

2. $y = x^2 - 6x + 7$
   $y = x^2 - 4x - 1$

$$\cancel{x^2} - 6x + 7 = \cancel{x^2 - 4x - 1}$$
$$-\cancel{x^2} + 4x + 1 \qquad -x^2 + 4x + 1$$

$$-2x + 8$$
$$-2x = -8$$
$$x = 4$$

$$y = (4)^2 - 4(4) - 1$$
$$y = 16 - 16 - 1$$
$$y = -1$$



Solutions:
$(4,-1)$

3. $y = -2(x-3)^2 + 1$
$y = (x-3)^2 - 2$

$-2(x-3)^2+1 = (x-3)^2 - 2$

$-2(x^2-6x+9)+1 = x^2-6x+9-2$

$-2x^2+12x-17 = x^2-6x+7$

$+2x^2-12x+17 \quad +2x^2-12x+17$

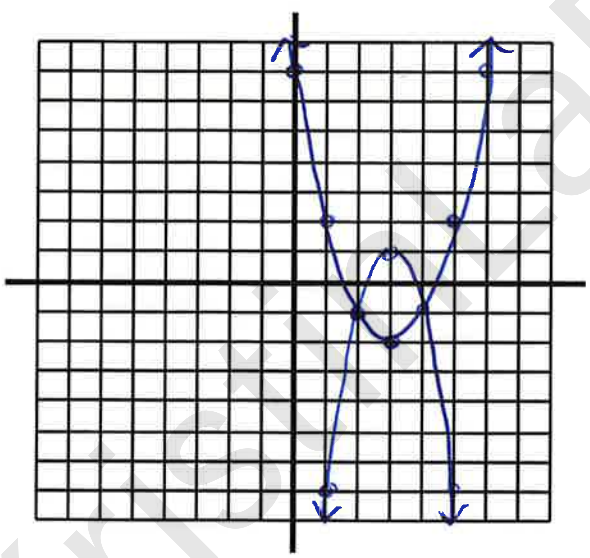
$3x^2-18x+24$

$3(x^2-6x+8)$

$3(x-2)(x-4)$

$x=2 \quad x=4$

$y=-1 \quad y=-1$



Solutions:
$(2,-1)$ + $(4,-1)$

4. $y = x^2 - 1$
$y = \frac{1}{3}(x-1)^2$

$3\left[x^2-1\right] = \left[\frac{1}{3}(x-1)^2\right]3$

$3x^2-3 = (x-1)^2$

$3x^2-3 = x^2-2x+1$

$+2x \quad -x^2 \quad -1 \quad -x^2+2x-1$

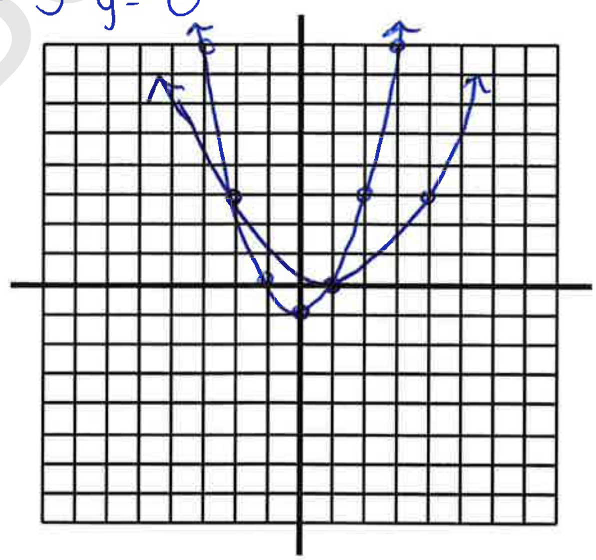
$2x^2+2x-4$

$2(x^2+x-2)$

$2(x+2)(x-1)$

$x=-2 \quad x=1$

$y=3 \quad y=0$



Solutions:
$(-2,3)$ + $(1,0)$

Solve the following systems using any method. Graph paper is attached for your use if you choose to solve graphically.

5. $x^2 + y^2 = 13$
   $y = x - 1$

$x^2 + (x-1)^2 = 13$

$x^2 + x^2 - 2x + 1 = 13$

$2x^2 - 2x - 12$

$2(x^2 - x - 6)$

$2(x-3)(x+2)$

$x = 3$	$x = -2$
$y = 2$	$y = -3$

6. $x^2 + y^2 - 16x + 39 = 0$
   $x^2 - y^2 - 9 = 0$

$x^2 + \cancel{y^2} - 16x + 39$

$+\ x^2 - \cancel{y^2} + 0x - 9$

$2x^2 - 16x + 30$

$2(x^2 - 8x + 15)$

$2(x-3)(x-5)$

$x = 3$	$x = 5$	$x = 5$
$y = 0$	$y = 4$	$y = -4$

7. $6x^2 + 3y^2 = 12$
   $y = -x + 2$

$6x^2 + 3(-x+2)^2 = 12$

$6x^2 + 3(x^2 - 4x + 4) = 12$

$6x^2 + 3x^2 - 12x + 12 = 12$

$9x^2 - 12x = 0$

$3x(3x - 4)$

$x = 0$	$x = 4/3$
$y = 2$	$y = 2/3$

8. $-3x + y = 6$
   $8x + y^2 + 24 = 0$

$y = 6 + 3x$

$8x + (6+3x)^2 + 24 = 0$

$8x + 36 + 36x + 9x^2 + 24$

$9x^2 + 44x + 60$

$\boxed{\text{no solutions}}$

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FORMATIVE ASSESSMENT: SYSTEMS OF EQUATIONS

SHOW ALL WORK. Write your answers on the lines provided.

Solve each system algebraically using either the substitution or elimination method. Then graph each system to verify your solution. {6 pts each}

1. $y = -x^2 - 3x - 4$
   $y = -x + 2$

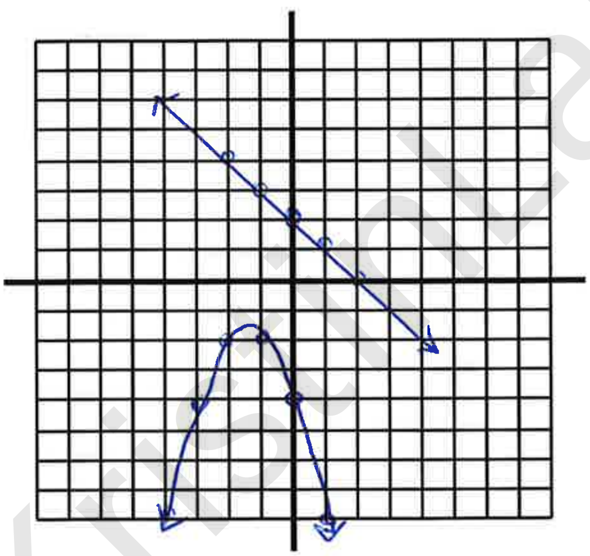
$y + x^2 + 3x = -4$

$-y + 0x^2 - x = -2$

$x^2 + 2x = -6$

$x^2 + 2x + 6$

PRIME



Solutions:

NO SOLUTIONS

2. $y = x^2 + 4$
   $y = x + 4$

$x + 4 = x^2 + 4$

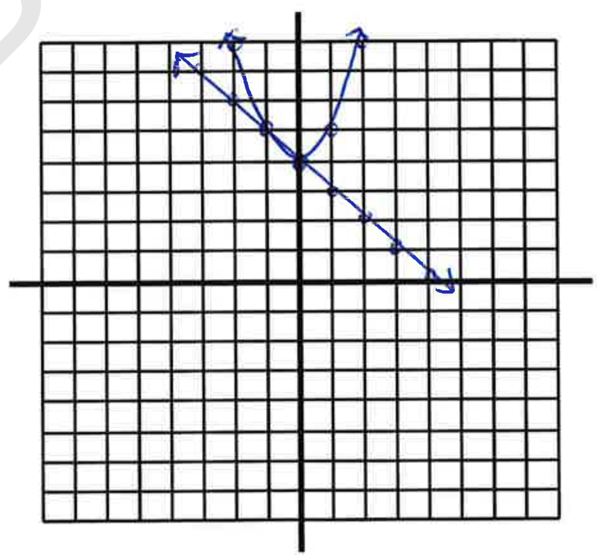
$-x - 4 \quad -4 - x$

$x^2 - x$

$x(x-1)$

$x = 0 \quad x = 1$

$y = 4 \quad y = 5$



Solutions:

$(0,4)$ + $(1,5)$

Solve the following two linear systems using Cramer's Rule. {4 pts each}

3. $2x - y = -2$
   $x + 2y = 14$

$\boxed{2, 6}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} = 4 - -1 = 5$$

$$x = \frac{\begin{bmatrix} -2 & -1 \\ 14 & 2 \end{bmatrix}}{5} = \frac{-4 - -14}{5} = \frac{10}{5} \qquad y = \frac{\begin{bmatrix} 2 & -2 \\ 1 & 14 \end{bmatrix}}{5} \quad \frac{28 - -2}{5} = \frac{30}{5}$$

$x = 2$ $\qquad$ $y = 6$

4. $4x + y + 3z = 7$
   $2x - 5y + 4z = -19$
   $x - y + 2z = -2$

$\boxed{-1, 5, 2}$

$$\begin{bmatrix} 4 & 1 & 3 \\ 2 & -5 & 4 \\ 1 & -1 & 2 \end{bmatrix} \begin{matrix} 4 & 1 \\ 2 & -5 \\ 1 & -1 \end{matrix} = (-40 + 4 + -6) - (-15 + (-16) + 4)$$

$$-42 \quad - \quad -27$$

$D: -15$

USING CALC

$$x = \frac{15}{-15} = -1 \qquad y = \frac{-75}{-15} = 5 \qquad z = \frac{-30}{-15} = 2$$